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On Ky Fan's Section Theorem*

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In this paper we first give a new section theorem of Ky Fan type and its equivalent theorem in an H-space, and next as their applications we obtain some minimax theorems and a coincidence theorem. © 1998 Academic Press

Key Words: H-space; H-convex set; upper (lower) semicontinuous function; section theorem; multivalued mapping.

1. INTRODUCTION AND PRELIMINARIES

In 1961, Ky Fan [3] proved the following section theorem:

THEOREM A. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E, and let A be a nonempty subset of $X \times X$ with $(x, x) \in A$ for all $x \in X$. Suppose the following conditions are fulfilled*:

(i) *For each $x \in X$, the set $\{y \in X: (x, y) \notin A\}$ is convex or empty.*

(ii) *For each $y \in X$, the set $\{x \in X: (x, y) \in A\}$ is closed.*

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset A$.

In 1980, Ha [5] generalized Theorem A as follows:

THEOREM B. *Let E and F be Hausdorff topological vector spaces, let $X \subset E$, $Y \subset F$ be nonempty convex subsets, and let A be a subset of $X \times Y$ such that*

(i) *For each $x \in X$, the set $\{y \in Y: (x, y) \notin A\}$ is convex or empty.*

(ii) *For each $y \in Y$, the set $\{x \in X: (x, y) \in A\}$ is closed in X.*

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Suppose that there exists a subset B of A and a compact convex subset K of X such that B is closed in $X \times Y$ and

(iii) *For each $y \in Y$, the set $\{x \in K: (x, y) \in B\}$ is nonempty and convex. Then there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset A$.*

This paper has two purposes. First we give a section theorem of the Ky Fan type and its equivalent theorem in an H-space, and next, as their applications, we obtain some minimax theorems and some coincidence theorems.

To begin with we explain the notion of an H-space introduced by Horvath [6–8] and related concepts on H-spaces.

Let X be a topological space and let $\mathscr{F}(X)$ be the family of all nonempty finite subsets of X. Let $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in \mathscr{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space. Given an H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called H-convex if $\Gamma_A \subset D$ for each nonempty finite subset A of D. For a nonempty subset K of X, we define the H-convex hull of K, denoted by H-*co* K, as

$$H\text{-}co\, K = \bigcap \{ D \subset X \colon D \text{ is } H\text{-convex and } D \supset K \}.$$

Then H-*co* K is H-convex and is the smallest H-convex set containing K.

Let $(X, \{\Gamma_A\})$ be an H-space, let Y be a topological space, and let f: $X \times Y \to R$ (real number set) be a function. For each $y \in Y$, $f(x, y)$ is said to be H-quasiconvex (or H-quasiconcave) in x if the set $\{x \in X: f(x, y) < t\}$ (or $\{x \in X: f(x, y) > t\}$) is H-convex for all $t \in R$.

Now let X, Y be two topological spaces.

(1) If $A \subset X \times Y$ and $y \in Y$, let $A[y] = \{x \in X: (x, y) \in A\}$.

(2) For a multivalued mapping T: $X \to 2^Y$, we denote $T^{-1}(y) = \{x \in X: y \in T(x)\}$ and $T^*(y) = X \setminus T^{-1}(y)$ for each $y \in Y$.

(3) A function f: $X \times Y \to R$ is said to be transfer lower (upper) semicontinuous about x on X if for each $r \in R$, and each $x \in X$ and $y \in Y$, $f(x, y) > r$ $(f(x, y) < r)$ implies that there exists an open neighborhood $N(x)$ of x and a point $y' \in Y$ such that $f(z, y') > r$ $(f(z, y') < r)$ for all $z \in N(x)$ [17].

Remark. (a) If f is lower (upper) semicontinuous in x, then f is transfer lower (upper) semicontinuous about x on X.

(b) If function f: $X \times Y \to R$ is transfer lower (upper) semicontinuous about x on X, then the marginal function $x \to \sup_{y \in Y} f(x, y)$ is lower semicontinuous ($x \to \inf_{y \in Y} f(x, y)$ is upper semicontinuous). The converse is not true.

EXAMPLE. Let $X = Y = [0, 1]$. Suppose the function $f\colon X \times Y \to R$ is defined by

$$f(x, y) = \begin{cases} D(x) D(y), & \text{if } x \neq 1 \text{ and } y \neq 1. \\ 1 - D(y), & \text{if } x = 1, \\ 1 - D(x), & \text{if } y = 1, \end{cases}$$

where $D(x)$ is the Dirichlet function on $[0, 1]$. Then

$$\sup_{y \in Y} f(x, y) = 1, \qquad \inf_{y \in Y} f(x, y) = 0.$$

Obviously both of the marginal functions

$$x \to \sup_{y \in Y} f(x, y)$$

and

$$x \to \inf_{y \in Y} f(x, y)$$

are continuous in X, but $f(x, y)$ is neither transfer lower semicontinuous nor transfer upper semicontinuous about x on X.

2. MAIN RESULTS

Lemmas 1 and 2 are, respectively, Theorem 2.2 and Corollary 2.3 in Tarafdar [16].

LEMMA 1. *Let X be a compact topological space and let $(Y, \{\Gamma_A\})$ be an H-space. Let $T\colon X \to 2^Y$ be a multivalued mapping such that*

(i) *For each $x \in X$, $T(x)$ is a nonempty H-convex subset of Y.*

(ii) *$\{\operatorname{int}(T^{-1}(y))\colon y \in Y\}$ is an open covering of X.*

Then there is a continuous selection $f\colon X \to Y$ of T such that $f = g \circ \varphi$, where $g\colon \Delta_n \to Y$ and $\varphi\colon X \to \Delta_n$ are continuous mappings, and Δ_n is the standard n-dimensional simplex for some positive integer n.

LEMMA 2. *Let $(X, \{\Gamma_A\})$ be a compact H-space and $T\colon X \to 2^X$ be a multivalued mapping such that*

(i) *For each $x \in X$, $T(x)$ is a nonempty H-convex subset.*

(ii) *$\{\operatorname{int}(T^{-1}(y))\colon y \in Y\}$ is an open covering of X.*

Then there is a point $x_0 \in T(x_0)$.

THEOREM 1. *Let X be a Hausdorff topological space, let $(Y,\{\Gamma_B\})$ be an H-space, and let M, N be two subsets of $X \times Y$. Suppose the following conditions are fulfilled*:

(i) *There exists a compact subset K of X such that*

(a) *for each $x \in K$, the set*

$$H - co\{y \in Y\colon (x,y) \notin N\} \subset \{y \in Y\colon (x,y) \notin M\};$$

(b) $\bigcap_{y \in Y} cl(N[y]) = \bigcap_{y \in Y} N[y]$.

(ii) *There exists a subset P of M such that P is closed in $X \times Y$, and for each $y \in Y$, the set*

$$\{x \in K\colon (x,y) \in P\}$$

is nonempty acyclic.

Then there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset N$.

Proof. If the conclusion of Theorem 1 is false, then for each $x \in K$ there is a point $y_0 \in Y$ such that $(x, y_0) \notin N$. Let

$$S(x) = \{y \in Y\colon (x,y) \notin N\}, \qquad T(x) = \{y \in Y\colon (x,y) \notin M\}.$$

Then $S, T\colon K \to 2^Y$ are such that for each $x \in K$, $S(x) \neq \emptyset$ and H-$co\,S(x) \subset T(x)$ by (i)(a). For each $x \in X$, there exists a point $y \in S(x)$, i.e., $x \notin N[y]$ since $S(x) \neq \emptyset$, and hence there exists a point $y' \in Y$ such that $x \notin cl(N[y'])$ by (i)(b). Consequently, there is an open neighborhood $U(x)$ of x such that $U(x) \cap N[y'] = \emptyset$. Hence $U(x) \subset S^{-1}(y')$, i.e., $x \in \operatorname{int}(S^{-1}(y'))$. Therefore, $\{\operatorname{int}(S^{-1}(y))\colon y \in Y\}$ is an open covering of K. Consequently, $\{\operatorname{int}(H\text{-}co\,S)^{-1}(y)\colon y \in Y\}$ is an open covering of K, where the mapping H-$co\,S\colon K \to 2^Y$ is defined by

$$H\text{-}co\,S(x) = H\text{-}co(S(x)), \qquad \forall x \in K.$$

By virtue of Lemma 1 there exists a continuous mapping $f\colon K \to Y$ such that $f = g \circ \psi$, and

$$f(x) \in H\text{-}co\,S(x) \subset T(x)$$

for all $x \in K$, where $\psi\colon K \to \Delta_n$, $g\colon \Delta_n \to Y$ are continuous mappings and Δ_n is the standard n-simplex. Hence

$$(x, f(x)) \notin M, \qquad \forall x \in K. \tag{1}$$

On the other hand, we define a multivalued mapping $G\colon Y \to 2^K$ by

$$G(y) = \{x \in K\colon (x, y) \in P\}, \qquad \forall y \in Y.$$

By (ii), $G\colon Y \to 2^K$ is an upper semicontinuous multivalued mapping with nonempty closed acyclic values. Consequently, so is the mapping $F\colon \Delta_n \to 2^K$ defined by $F(u) = G(g(u))$. By virtue of Lemma 2.1 of [13], there exists a point $\bar{u} \in \Delta_n$ such that $\bar{u} \in \psi(F(\bar{u})) = \psi[G(g(\bar{u}))]$, and so there is a point $\bar{x} \in G(g(\bar{u})) \subset K$ such that $\bar{u} = \psi(\bar{x})$. Let $\bar{y} = g(\bar{u})$. Then $\bar{y} = g \circ \psi(\bar{x}) = f(\bar{x})$ and $\bar{x} \in G(\bar{y})$, i.e., $(\bar{x}, f(\bar{x})) = (\bar{x}, \bar{y}) \in P \subset M$. This contradicts (1) and completes the proof.

COROLLARY 2. *Let X be a Hausdorff topological space, let $(Y, \{\Gamma_B\})$ be an H-space, and let M, N be two subsets of $X \times Y$ with $M \subset N$. Suppose the following conditions are fulfilled*:

(i) *For each $y \in Y$, the set $\{x \in X\colon (x, y) \in N\}$ is closed.*

(ii) *For each $x \in X$, the set $\{y \in Y\colon (x, y) \notin M\}$ is H-convex or empty.*

Suppose also that there exists a subset P of M and a compact subset K of X such that P is closed in $X \times Y$ and

(iii) *For each $y \in Y$, the set $\{x \in K\colon (x, y) \in P\}$ is nonempty acyclic.*

Then there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset N$.

Remark. Both Theorem 1 and Corollary 2 improve and extend Theorem A and Theorem B to a topological space X and an H-space Y.

Theorem 3 is equivalent to Theorem 1.

THEOREM 3. *Let X be a Hausdorff topological space, let $(Y, \{\Gamma_B\})$ be an H-space, and let F, $T\colon Y \to 2^X$ be two multivalued mappings. Suppose the following conditions are fullfilled*:

(i) $\bigcup_{y \in Y} F^{*-1}(y) = \bigcup_{y \in Y} \operatorname{int}(F*^{-1}(y))$.

(ii) *$T\colon Y \to 2^X$ is an upper semicontinuous multivalued mapping with nonempty closed values (or T has a closed graph).*

(iii) *There exists a compact subset K of X such that*

(a) *for each $y \in Y$, the set $T(y) \cap K$ is nonempty acyclic,*

(b) *For each $x \in K$, H-co$(F^*(x)) \subset T^*(x)$.*

Then $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Proof. Theorem 1 ⇒ Theorem 3: Let

$$N = \{(x, y) \in X \times Y: x \in F(y)\},$$
$$M = \{(x, y) \in X \times Y: x \in T(y)\}.$$

Then for each $x \in K$, the set

$$\begin{aligned} H\text{-}co\{y \in Y: (x, y) \notin N\} &= H\text{-}co\big(Y \setminus F^{-1}(x)\big) \\ &\subset Y \setminus T^{-1}(x) \\ &= \{y \in Y: (x, y) \notin M\} \end{aligned}$$

by (iii)(b). By (i) $\bigcap_{y \in Y} cl(N[y]) = \bigcap_{y \in Y} N[y]$, by (ii) M is closed, and by (iii)(a) the set

$$\{x \in K: (x, y) \in M\} = K \cap T(y)$$

is nonempty acyclic for each $y \in Y$.

By Theorem 1, there exists a point $x_0 \in K$ such that $\{x_0\} \times Y \subset N$, i.e., $(x_0, y) \in N$ for all $y \in Y$. Hence $x_0 \in \bigcap_{y \in Y} F(y) \neq \emptyset$. This completes the proof.

Theorem 3 ⇒ Theorem 1: For each $y \in Y$, let

$$S(y) = \{x \in X: (x, y) \in M\},$$
$$T(y) = \{x \in X: (x, y) \in P\},$$
$$F(y) = \{x \in X: (x, y) \in N\}.$$

By (i) there exists a compact subset $K \subset X$ such that $K \cap T(y)$ is nonempty acyclic and for each $x \in K$ the following hold:

(a) $H\text{-}co(F^*(x)) \subset S^*(x) \subset T^*(x)$.

(b) $\bigcup_{y \in Y} \operatorname{int}(F*^{-1}(y)) = \bigcup_{y \in Y} F*^{-1}(y)$.

Since again P is closed in $X \times Y$, T has a closed graph. Consequently by Theorem 3 $\bigcap_{y \in Y} F(y) \neq \emptyset$, and hence there is a point $x_0 \in \bigcap_{y \in Y} F(y)$. Thus $\{x_0\} \times Y \subset N$ and Theorem 1 is proved.

COROLLARY 4. *Let X be a Hausdorff topological space, let $(Y, \{\Gamma_B\})$ be an H-space, and let $F, T: Y \to 2^X$ be two multivalued mappings. Suppose the following conditions are fulfilled*:

(i) $\bigcup_{y \in Y} F^{*-1}(y) = U_{y \in Y} \operatorname{int}(F^{*-1}(y))$.

(ii) *$T: Y \to 2^X$ is an upper semicontinuous multivalued mapping with nonempty closed values (or T has a closed graph),*

(iii) *There exists a compact subset K of X such that*

(a) *for each* $y \in Y$, *the set* $T(y) \cap K$ *is nonempty acyclic and* $T(y) \subset F(y)$,

(b) *for each finite subset* $\{y_1, y_2, \dots, y_n\} \subset Y$,

$$T\left(\Gamma_{\{y_1, y_2, \dots, y_n\}}\right) \subset \bigcup_{i=1}^{n} T(y_i).$$

Then $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Proof. Since $T(y) \subset F(y)$ for each $y \in Y$, $T^{-1}(x) \subset F^{-1}(x)$ for each $x \in X$, and hence $F^*(x) \subset T^*(x)$ for each $x \in X$. Hence, it is sufficient to show that $T^*(x)$ is H-convex. For each finite subset $\{y_1, y_2, \dots, y_n\} \subset T^*(x)$, we have $x \notin \bigcup_{i=1}^{n} T(y_i)$, and hence $x \notin T(\Gamma_{\{y_1, y_2, \dots, y_n\}})$. Therefore, for each $y \in \Gamma_{\{y_1, y_2, \dots, y_n\}}$, we have $x \notin T(y)$, i.e., $y \in T^*(x)$. Hence $\Gamma_{\{y_1, y_2, \dots, y_n\}} \subset T^*(x)$. This shows that $T^*(x)$ is H-convex and completes the proof.

Remark. Even if X *and* Y are nonempty convex setsets of topological vector spaces and $F = T$, Theorem 3 and Corollary 4 are different from Theorems 2.4 and 2.5 in [19], since our multivalued mapping is of upper semicontinuous type.

THEOREM 5. *Let X be a compact Hausdorff topological space, let* $(Y, \{\Gamma_A\})$ *be an H-space, and let* $f, g\colon X \times Y \to R$ *be two functions. Let* $\beta = \sup_{x \in X} \inf_{y \in Y} g(x, y)$ *and* $D = \{(x, y) \in X \times Y\colon f(x, y) > \beta\}$. *Suppose that the following conditions are fulfilled*:

(i) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in D$.

(ii) $g(x, y)$ *is transfer upper semicontinuous in* x, *and for each* $y \in Y$, *the set* $\{x \in X\colon f(x, y) > t\}$ *is acyclic or empty for each* $t > \beta$.

(iii) *For each* $x \in X$, $f(x, y)$ *is H-quasiconvex in* y, *and* $f\colon X \times Y \to R$ *is upper semicontinuous. Then*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. If the conclusion of Theorem 5 is false, then there is a real number t such that

$$\beta = \sup_{x \in X} \inf_{y \in Y} g(x, y) < t < \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

For each $y \in Y$, let $F(y) = \{x \in X\colon g(x, y) \geq t\}$, $T(y) = \{x \in X\colon f(x, y) \geq t\}$. Then $F, T\colon Y \to 2^X$ are two multivalued mappings with $T(y) \subset F(y)$ for all $y \in Y$, since $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$.

Since f is upper semicontinuous, the graph $\mathrm{Gr}(T)$ of T is closed, and since X is compact, $T: Y \to 2^X$ is upper semicontinuous. Since again, for each $x \in X$, $f(x, \cdot)$ is H-quasiconvex, the set

$$T^*(x) = \{y \in Y: x \notin T(y)\} = \{y \in Y: f(x,y) < t\}$$

is H-convex.

For each $x \in \bigcup_{y \in Y} F*^{-1}(y)$ there is a point $y_0 \in Y$ such that $g(x, y_0) < t$. By the first part of condition (ii), there exists an open neighborhood $U(x)$ of x and a point $y' \in Y$ such that $g(z, y') < t$ for all $z \in U(x)$, and hence $U(x) \cap F(y') = \varnothing$. Hence $x \in \mathrm{int}(F^{*-1}(y'))$. It shows

$$\bigcup_{y \in Y} F*^{-1}(y) = \bigcup_{y \in Y} \mathrm{int}\bigl(F^{*-1}(y)\bigr).$$

Note that since X is compact by the second part of (ii) and (iii) for each $y \in Y$, the set

$$T(y) = \{x \in X: f(x,y) \geq t\} = \bigcap_{0 < \varepsilon < t - \beta} \{x \in X: f(x,y) > t - \varepsilon\}$$

is acyclic (this follows from the continuity of Cech homology).

Therefore all conditions of Theorem 3 are fulfilled, so by Theorem 3, $\bigcap_{y \in Y} F(y) \neq \varnothing$. Thus there exists a point $x_0 \in X$ such that $x_0 \in \bigcap_{y \in Y} F(y)$, i.e., $g(x_0, y) \geq t$ for all $y \in Y$. Hence

$$\sup_{x \in X} \inf_{y \in Y} g(x,y) \geq t.$$

This contradicts the choice of t and completes the proof.

COROLLARY 6. *Let X be a compact Hausdorff topological space, and let $(Y, \{\Gamma_A\})$ be an H-space. If an upper semicontinuous function $f: X \times Y \to R$ is such that*

(i) *For each $y \in Y$, the set $\{x \in X: f(x,y) > t\}$ is acyclic or empty for each $t \in R$,*

(ii) *For each $x \in X$, $f(x,y)$ is H-quasiconvex in y, then*

$$\inf_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \inf_{y \in Y} f(x,y).$$

Proof. By Theorem 5 we have

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

Note that X is compact and f is upper semicontinuous, so we have

$$\inf_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\inf_{y\in Y} f(x,y).$$

This completes the proof.

Remark. In the case where X is compact, Corollary 6 contains Theorem 4 of Ha [5] as a special case.

THEOREM 7. *Let K be a nonempty compact subset of a Hausdorff topological space X, and let $(Y,\{T_A\})$ be an H-space. Let $T\colon X\to 2^Y$ and $F\colon Y\to 2^X$ be multivalued mappings such that*

(i) *F has a closed graph* $\mathrm{Gr}(F)$.

(ii) $\{\mathrm{int}(T^{-1}(y));\ y\in Y\}$ *is an open covering of X.*

(iii) *For each $x\in K$, $T(x)$ is nonempty H-convex.*

(iv) *For each $y\in Y$, $F(y)\cap K$ is nonempty acyclic.*

Then there exist a point $x_0\in X$ and a point $y_0\in Y$ such that $x_0\in F(y_0)$ and $y_0\in T(x_0)$.

Proof. For each $x\in X$, let $G(x)=F^{-1}(x)=\{y\in Y\colon x\in F(y)\}$. Then by (i) $G\colon X\to 2^Y$ has a closed graph and by (iv) for each $y\in Y$,

$$\begin{aligned} G^{-1}(y)\cap K &= \{x\in X\colon y\in G(x)\}\cap K\\ &= \{x\in X\colon x\in F(y)\}\cap K\\ &= F(y)\cap K \end{aligned}$$

is nonempty acyclic. Let $P=\mathrm{Gr}(G)$ and $M=N=\{(x,y)\in X\times Y\colon y\notin T(x)\}$. Then for each $y\in Y$, the set

$$\{x\in K\colon (x,y)\in P\} = \{x\in K\colon x\in G^{-1}(y)\} = K\cap G^{-1}(y)$$

is nonempty acyclic, and by (iii) P is closed in $X\times Y$. If the conclusion of Theorem 7 is false, then $T(x)\cap G(x)=\varnothing$ for all $x\in X$, and hence $P\subset M$. Moreover, by (iii) we know that for each $x\in K$, the set

$$\{y\in Y\colon (x,y)\notin M\} = T(x)$$

is H-convex, and by (ii) $\bigcap_{y\in Y} cl(N[y]) = \bigcap_{y\in Y} N[y]$.

So by Theorem 1, there exists a point $x_0\in K$ such that $\{x_0\}\times Y\subset M$, i.e., $y\notin T(x_0)$ for all $y\in Y$. Hence $T(x_0)=\varnothing$. This contradicts (iii) and completes the proof.

Remark. Theorem 7 improves Theorem 3.1 of [2], and improves and generalizes Theorem 2.3 of [12], Theorem 7 of [15], Theorem 1 of [9], Theorem 1 of [1], and Theorem 1 of [18].

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